Two methods for measuring Bell nonlocality via local unitary invariants of two-qubit systems in Hong-Ou-Mandel interferometers

Karol Bartkiewicz[1,2], Grzegorz Chimczak[1]

[1]Faculty of Physics, Adam Mickiewicz University in Poznań,
[2]RCPTM, Joint Laboratory of Optics of Palacký University and Institute of Physics of Academy of Sciences of the Czech Republic, Faculty of Science, Palacký University, Olomouc, Czech Republic

Mathematical framework

Two-qubit density matrix ρ (15 real parameters)

in Hilbert-Schmidt form

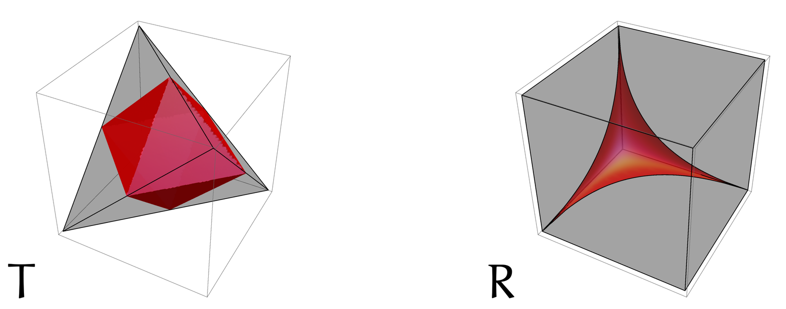
$$\rho = \tfrac{1}{4}(I\otimes I + \vec{x}\cdot\vec{\sigma}\otimes I + I\otimes\vec{y}\cdot\vec{\sigma} + \sum_{i,j=1}^{3} T_{i,j}\,\sigma_i\otimes\sigma_j),$$

where $\vec{\sigma} = [\sigma_1,\sigma_2,\sigma_3]$. The elements of the Bloch vectors are $x_i = \mathrm{Tr}[\rho(\sigma_i\otimes I)]$ and $y_i = \mathrm{Tr}[\rho(I\otimes\sigma_i)]$, respectively.

$$T_{i,j} = \mathrm{Tr}[\rho(\sigma_i\otimes\sigma_j)]$$

Can be diagonalized with only local operations (HWPs, QWPs).

Geometric pictures of physical states



Space of correlation matrix T (9 real parameters)

Each point is given as

$$\vec{t} = [T_{1,1}, T_{2,2}, T_{33}],$$

where T is the diagonalized correlation matrix. All physical states can be moved to the tetrahedron $\mathcal{T}$ (or $-\mathcal{T}$). All separable states are found in the octahedron [1].

Space of Horodecki matrix R (6 real parameters)

It is convenient to use real, positive and symmetric matrix $R = T^T T$ (6 real parameters). In this representation each physical state is now found in a cube.

Local two-qubit invariants

Makhlin's invariants [2, 3] (HOM-accessible)

$$\begin{aligned}
I_1 &= -\tfrac{8}{3}\{l_0[l_0(4l_0-3)+6(\bar{c}_1-2\bar{l}_1)] \\
&\quad +3\bar{l}_1-6\bar{c}_2+8\bar{l}_2\},\\
I_2 &= 1+16l_1-4(c_2+c_1),\\
I_3 &= 1+256\left(c_2^2+4c_3+c_1^2+l_2\right)-8(c_2+c_1),\\
I_4 &= 1-4c_2,\\
I_5 &= -4c_1+32c_3-64c_5+(1-4c_2)^2,\\
I_6 &= 1-1024c_8-4(3c_2+2c_1)\\
&\quad +16(3c_2^2+4c_3+2c_2c_1+c_1^2)\\
&\quad -64(c_2^3+4c_2c_3+2c_5+2c_3c_1+c_4)\\
&\quad +256(c_3^2+2c_2c_5+2c_6)\\
I_7 &= 1-4c_1,\\
I_8 &= -4c_2+32c_3-64c_4+(1-4c_1)^2,\\
I_9 &= 1-1024c_7-4(2c_2+3c_1)\\
&\quad +16(c_2^2+4c_3+2c_2c_1+3c_1^2)\\
&\quad -64(2c_2c_3+c_5+4c_3c_1+c_1^3+2c_4)\\
&\quad +256(c_3^2+2c_6+2c_1c_4),\\
I_{12} &= 1+16c_3-4(c_2+c_1),\\
I_{13} &= 1+256c_6-8(c_2+c_1)\\
&\quad +16(c_2^2+3c_3+c_2c_1+c_1^2)\\
&\quad -64[c_5+c_3(c_2+c_1)+c_4],\\
I_{14} &= 16[l_0^2(1-4\bar{c}_1)+2l_0(4\bar{c}_2-\bar{c}_1)\\
&\quad -\bar{l}_1+4\bar{c}_1\bar{l}_1+2\bar{c}_2-8\bar{c}_3]
\end{aligned}$$

Measuring the remaining Makhlin's invariants requires more measurements types than only simple singlet projections [3].

Jing's invariants [4, 3]

We find that Jing's invariants can be related to other Makhlin's invariants via singlet projections in the following way

$$\begin{aligned}
J_1 &= I_2\\
J_2 &= I_3\\
J_3 &= \tfrac{1}{2}(6I_1^2-I_2^3+3I_2I_3)\\
J_4 &= I_4,\quad J_5 = I_5,\quad J_6 = I_6,\quad J_7 = I_7,\\
J_8 &= I_8,\quad J_9 = I_9,\quad J_{10} = I_{12},\quad J_{11} = I_{13},
\end{aligned}$$

and $J_{12} = 4096c_9-1024(c_7+c_8+c_6c_1+c_6c_2+c_3c_4+c_3c_5)+768(3c_6+c_4c_1+c_5c_2+c_3^2)-64[2c_4+2c_5+6c_3(c_1+c_2)+c_1^3+c_2^3+c_1^2c_2+c_2^2c_1]+16(5c_3+3c_1^2+3c_2^2+4c_1c_2)-12(c_1+c_2)+1$. We can also express J_3 as $J_3 = 4096l_3-3072(c_7+c_8)+768(2c_6+c_4c_1+c_5c_2+c_3^2)-64[3c_4+3c_5+6c_3(c_1+c_2)+c_1^3+c_2^3]+48(2c_3+c_1^2+c_2^2+c_1c_2)-12(c_1+c_2)+1.$

Measuring nonlocality

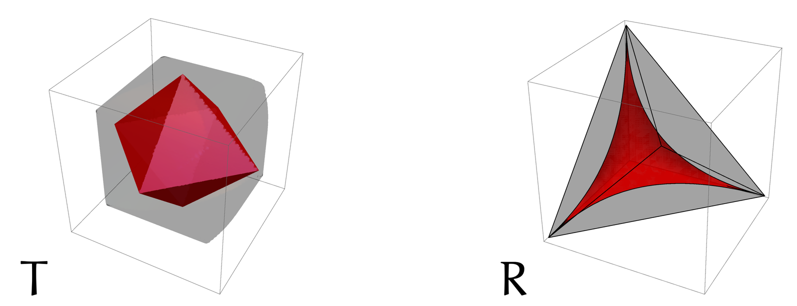
Bell CHSH inequality

For local states

$$\max_{\mathcal{B}_{\mathrm{CHSH}}} |\mathrm{Tr}\,(\rho\,\mathcal{B}_{\mathrm{CHSH}})| = 2\sqrt{\mathrm{Tr}R-\min[\mathrm{eig}(R)]} \le 2,$$

where $\mathcal{B}_{\mathrm{CHSH}} = \hat{a}\cdot\vec{\sigma}\otimes(\hat{b}+\hat{b}')\cdot\vec{\sigma} + a'\cdot\vec{\sigma}\otimes(\hat{b}-\hat{b}')\cdot\vec{\sigma}$ depends on unit vectors in 3D real space $\hat{a}, \hat{b}, \hat{a}', \hat{b}'$ [5]. For correlations that do not break Bell's (CHSH) inequality (insecure QKD) one can construct a local hidden variable model [6, 7].

Geometric pictures of local states



CHSH nonlocality measure

$$M = \max\,[\mathrm{Tr}R-\min[\mathrm{eig}(R)]-1, 0]$$

It can be used for entanglement measure estimation [8].

Invariant-based eigenvalues of R (3 invariants)

The spectrum of a three-dimensional matrix is given by the roots of the following polynomial in r in terms of J-invariants [4, 3]

$$-r^3+J_1r^2+(J_1^2-J_2)r+J_1^3+2J_3-3J_1J_2 = 0$$

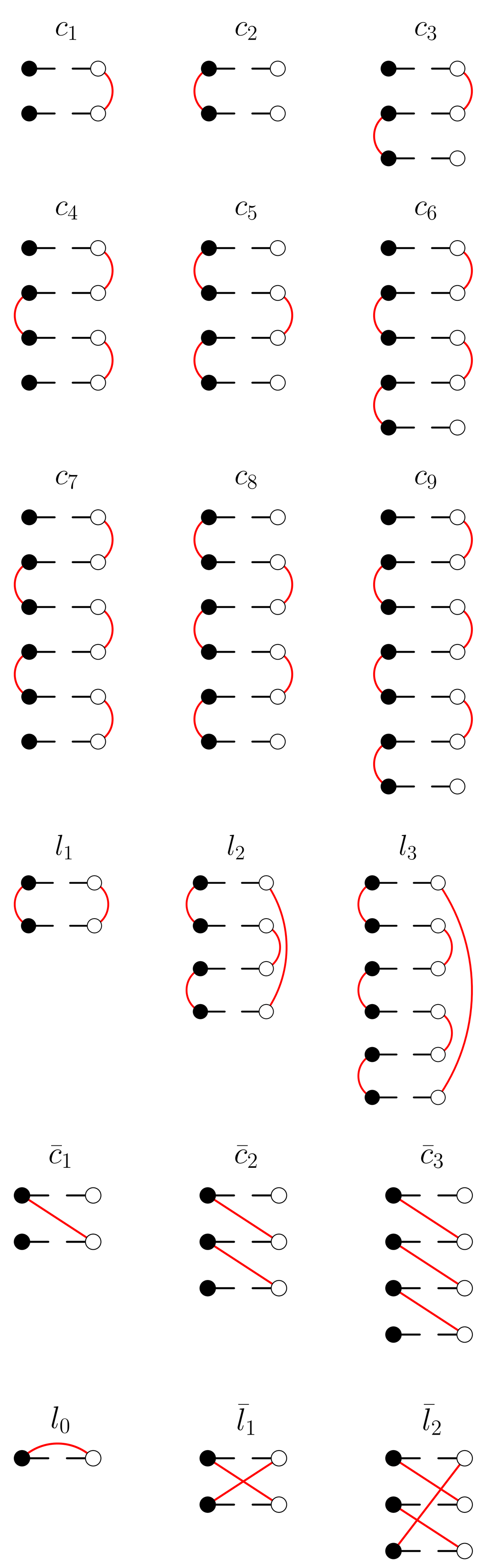
or in terms of I-invariants [2, 3]

$$-r^3+I_2r^2+(I_2^2-I_3)r+I_2^3+(6I_1^2-I_2^3) = 0.$$

Web of singlet projections

Prime detection events (singlet projections)

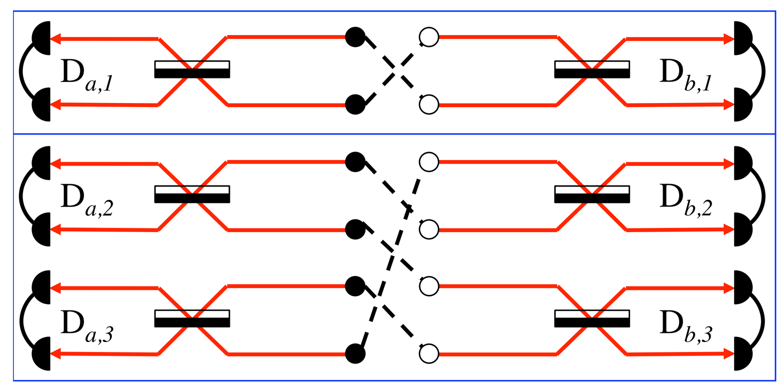
Events are found with local invariants and Cayley-Hamilton theorem [9, 3].



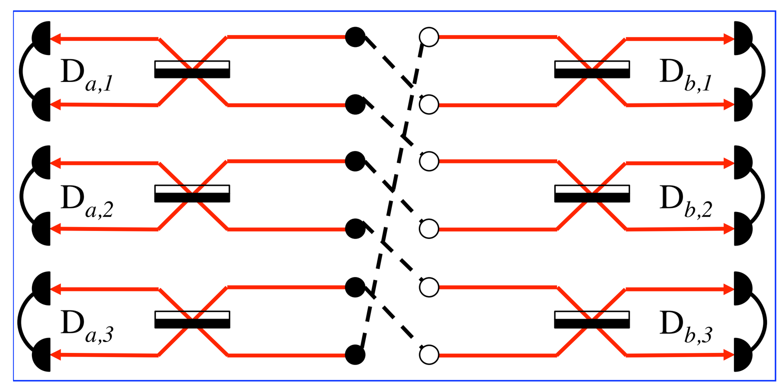
Experimental considerations

Local and nonlocal HOM interferometers for polarization-encoded qubits

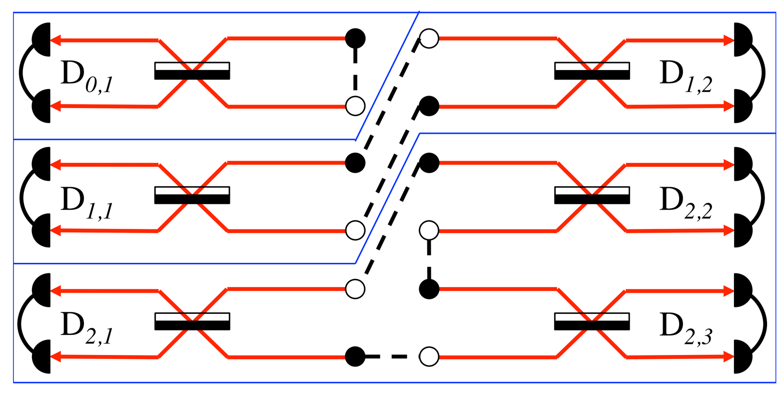
I_2, I_3 (or J_1, J_2)



I_1



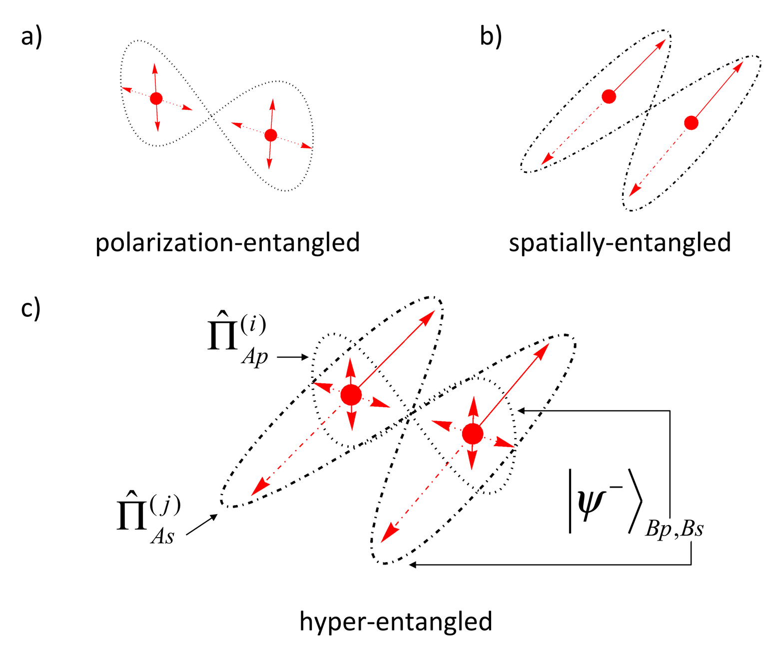
J_3



Measurement optimization

An alternative to multicopy measurement

We can use N degrees of freedom (hyperentanglement) [10] instead of N copies [11, 12].



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Acknowledgments

Grantová agentura České republiky
Designing and implementing multi-photon quantum circuits
Registry No.: 15-07559Y